JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 4, July-August 1992

Robust Control Systems Design Using H^∞ Optimization Theory

X. P. Li* and B. C. Chang†
Drexel University, Philadelphia, Pennsylvania 19104
and
S. S. Banda‡ and H. H. Yeh§
Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Dayton, Ohio 45433

In this paper, we show step-by-step procedures for applying the H^∞ theory to robust control systems design. The objective of the paper is to eliminate the possible difficulties a control engineer may encounter in applying H^∞ control theory. We will review the physical meanings of the H^∞ norm, explain how it relates to robustness issues, and show how to formulate H^∞ optimization problems, including the construction of a state-space realization of the generalized plant. An efficient algorithm is used to compute the optimal H^∞ norm. The controller formulas of Glover and Doyle are slightly modified and are used to construct an optimal controller without any numerical difficulty.

I. Introduction

It is well known that the H^∞ design technique[1-20] provides better robustness than the linear quadratic Gaussian (LQG), or H^2, method.[21,22] H^∞ control theory can handle the following two robustness issues: 1) minimization of the maximum error energy for all command/disturbance inputs with bounded energy and 2) closed-loop stability under unstructured plant uncertainties with a bounded H^∞ norm. H^∞ control theory is also indispensable in the structured singular value treatment of structured plant uncertainties.[23,24]

The elegant two-Riccati-equation approach proposed by Glover and Doyle (GD)[18] and Doyle, Glover, Khargonekar, and Francis (DGKF)[19] solved the standard H^∞ optimization problem. This approach characterizes all possible stabilizing suboptimal H^∞ controllers whose order is not higher than that of the generalized plant. However, there is not much information available in the literature that can give a detailed guideline in applying H^∞ theory to robust control design. The engineers may still encounter difficulties one way or another in applying the H^∞ control theory. These difficulties include the formulation of a robust control problem as an H^∞ optimization problem, the construction of a state-space realization of the generalized plant, the computation of the optimal H^∞ norm, and the construction of an optimal (or suboptimal) controller.

The objective of the paper is to eliminate the possible difficulties and provide a design procedure by which a control engineer can easily formulate and solve H^∞ optimization problems. We will review the physical meanings of the H^∞ norm, explain how it relates to the robustness issues mentioned earlier, and show how to formulate H^∞ optimization problems, including the construction of a state-space realization of the generalized plant. We will also introduce a fast algorithm that we recently developed for the computation of optimal H^∞ norm. In this algorithm the convex properties of H^∞ Riccati solutions are employed.[25] Because of the convexity, the convergence of the algorithm is guaranteed and the convergence rate is quadratic. In most situations there is a numerical difficulty when one uses GD controller formulas[18] to construct an optimal controller. However, as mentioned in Refs. 19 and 20, a descriptor (or generalized state-space representation[26]) version of the controller formulas can avoid this numerical difficulty. With slight rearrangement, GD controller formulas can be rewritten in a descriptor form, which can then be reduced to a lower order state-space representation.

The preliminaries, including the basic concept of the H^∞ norm, robust stability and GD formulas, are briefly reviewed in Sec. II. In Sec. III, we explain how to formulate a robust control problem as a two-block or four-block H^∞ optimization problem and how to construct a state-space realization of the generalized plant. An efficient algorithm to compute the optimal H^∞ norm and a procedure to construct an optimal H^∞ controller are both briefly described in Sec. IV. Some illustrative examples are included in Sec. V. Section VI is the conclusion.

II. Preliminaries

Throughout the paper, both of the notations

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and $\{A, B, C, D\}$ are used for the same purpose to represent a state-space realization of a system whose transfer function is $C(sI-A)^{-1}B+D$. $R(s)^{p\times q}$ is the set of $p\times q$ proper rational matrices with real coefficients. $(RH^\infty)^{m\times r}$ is the set of $m\times r$ proper rational matrices with real coefficients that are analytic in the closed right half plane. $(RL^\infty)^{m\times r}$ is the set of $m\times r$ proper rational matrices with real coefficients that are analytic on the imaginary axis.

H^∞ Norm of a System and its Physical Meanings

H^∞ control theory deals with the behaviors of systems with a certain set of inputs. The set of inputs, denoted U, consists of square (Lebesgue) integrable functions defined on all time $-\infty < t < \infty$ and taking values in R^r, i.e.,

$$\forall u(t) \in U, \qquad \int_{-\infty}^{\infty} u(t)^T u(t)\,\mathrm{d}t < \infty$$

Received Jan. 29, 1991; revision received Sept. 20, 1991; accepted for publication Oct. 2, 1991.

*Graduate Student, Department of Mechanical Engineering and Mechanics.

†Associate Professor, Department of Mechanical Engineering and Mechanics. Member AIAA.

‡Aerospace Engineer, Control Dynamics Branch. Associate Fellow AIAA.

§Electronics Engineer, Control Dynamics Branch.

It can be proven that U is a linear space. Moreover, L_2 norm can be defined on U to measure the "size" of each element in U:

$$\|u(t)\|_2 = \left\{\int_{-\infty}^{\infty} u(t)^T u(t)\,\mathrm{d}t\right\}^{1/2} \tag{1}$$

Let $\mathcal{G}$ be a stable system with the transfer matrix $G(s) = \{A,B,C,D\}$, and denote Y as the output signal space of $\mathcal{G}$ due to the input U, then Y is also a linear space with L_2 norm defined on it. From a mathematical point of view, $\mathcal{G}$ can be considered as a linear operator: $U \to Y$. To characterize the system $\mathcal{G}$, one needs to investigate the induced operator norm of $\mathcal{G}$, which is defined as[27]

$$\|\mathcal{G}\| := \sup\left\{\frac{\|y(t)\|_2}{\|u(t)\|_2} : u(t) \in U\right\}$$
$$= \sup\{\|y(t)\|_2 : u(t) \in U \quad \text{and} \quad \|u(t)\|_2 = 1\} \tag{2}$$

$\|\mathcal{G}\|$ generally characterizes the "transferability" for system $\mathcal{G}$, which measures the "size" of the output for a given input.

It can be proven[28] that the induced operator norm defined in Eq. (2) is identical to the H^∞ norm of its transfer matrix $G(s)$, i.e.,

$$\|\mathcal{G}\| = \|G(s)\|_\infty := \sup_\omega \bar{\sigma}[G(j\omega)] \tag{3}$$

Hence, one could compute the induced operator norm by computing the H^∞ norm of its transfer matrix.

By use of the preceding definitions, a physical interpretation of the H^∞ norm of a transfer matrix $G(s)$ can be given as follows. Recall that the energy of a signal $x(t)$ is defined as $\|x(t)\|_2^2$ (Ref. 29). Therefore, one can see that U consists of all inputs with bounded energy. Thus, the H^∞ norm of $G(s)$ is the maximal output energy for all inputs in U with unit energy.

The preceding interpretation can be extended a little further. For convenience, we refer to $u(t)$ in U as "energy signals." Now let us consider another set of input signals that we call "power signals" and denote U_p. For each $u(t) \in U_p$,

$$R_{uu}(\tau) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t+\tau)u^T(\tau)\,\mathrm{d}t \tag{4}$$

exists and is finite for all τ.

Since the power of signal $u(t)$ is[29]

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)^T u(t)\,\mathrm{d}t \tag{5}$$

then we can see that U_p consists of inputs with bounded power. Furthermore, a seminorm, denoted the "power norm," can be defined on both U_p and its output space Y_p:

$$\|u(t)\|_p := \left\{\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u(t)^T u(t)\,\mathrm{d}t\right\}^{1/2} \tag{6}$$

The power norm is a seminorm instead of a norm because $\|u(t)\|_P = 0$ does not imply $u(t) = 0$. For instance, any $u(t) \neq 0$ in U is also in U_p, but $\|u(t)\|_P = 0$. Similarly, we have output space Y_p with $\|\cdot\|_p$ defined on it. On these power normed spaces, the operator norm for $\mathcal{G}$ can be induced similarly:

$$\|\mathcal{G}\| := \sup\left\{\frac{\|y(t)\|_P}{\|u(t)\|_P} : u(t) \in U_P\right\}$$
$$= \sup\{\|y(t)\|_P : u(t) \in U_P \quad \text{and} \quad \|u(t)\|_P = 1\} \tag{7}$$

Although in general the induced operator norm depends on the norms defined on input space and output space, $\|\mathcal{G}\|$ in Eq. (7) is again identical to the H^∞ norm of $G(s)$. Therefore, the physical meaning of the H^∞ norm, when inputs and outputs are power signals, can be stated as the maximal output power for all inputs with unit power.

The third set of input signals are the spectral density signals, denoted U_s and consisting of signals with bounded spectral density; i.e., for any $u(t)$ in U_s,

$$\|S_{uu}(j\omega)\|_\infty := \sup_\omega \bar{\sigma}[S_{uu}(j\omega)] < \infty \tag{8}$$

where

$$S_{uu}(j\omega) := \int_{-\infty}^{\infty} R_{uu}(\tau)e^{-j\omega\tau}\,\mathrm{d}t \tag{9}$$

is the spectral density matrix and $R_{uu}(\tau)$ is defined in Eq. (4). Similarly, a seminorm, denoted the "spectral norm," can be defined both on U_s and its output space Y_s:

$$\|u(t)\|_s := \{\|S_{uu}(j\omega)\|_\infty\}^{1/2} \tag{10}$$

On these spectral density normed spaces, the operator norm of $\mathcal{G}$ can be defined:

$$\|\mathcal{G}\| := \sup\left\{\frac{\|y(t)\|_s}{\|u(t)\|_s} : u(t) \in U_s\right\}$$
$$= \sup\{\|y(t)\|_s : u(t) \in U_s \quad \text{and} \quad \|u(t)\|_s = 1\} \tag{11}$$

Again, it can be shown that

$$\|\mathcal{G}\| = \|G(s)\|_\infty \tag{12}$$

Hence, the H^∞ norm of a system indicates the maximal output "spectral density" for all inputs with unit "spectral density," when inputs and outputs are spectral density signals.

Uncertain Disturbances/Commands Attenuation

Now let us consider the disturbance attenuation problem with the following system:

$$y(s) = P(s)u(s) + v(s) \tag{13a}$$

$$u(s) = K(s)y(s) \tag{13b}$$

where $v(s)$ is the disturbance, $y(s)$ the output, and $K(s)$ the controller to be designed. The transfer function from $v(s)$ to $y(s)$ is given by

$$[I - P(s)K(s)]^{-1} \tag{14}$$

which is the sensitivity function of the closed-loop system. The H^∞ norm of the sensitivity function is the square root of the maximal energy of the disturbance response. One of the objectives of feedback control is to choose a controller $K(s)$ that stabilizes the closed-loop system and makes $\|W_1(I-PK)^{-1}\|_\infty$ as small as possible subject to control-input constraints. $W_1(s)$ is a weighting matrix used to emphasize the disturbance response over a certain frequency band.

Robust Stability and Control-Input Constraints

In the following the robust-stability and control-input constraints will be considered. We denote the unperturbed (nominal) closed-loop system in Eq. (13) by $\mathcal{C}(P,K)$ and the system with perturbed plant by $\mathcal{C}(\tilde{P},K)$. We shall consider three types of unstructured plant uncertainties: 1) additive perturbations, 2) multiplicative perturbations introduced at the output, and 3) multiplicative perturbations introduced at the input. Under these perturbations the perturbed plants are confined to a neighborhood of the nominal plant with certain magnitude.

Additive Perturbations

The perturbed plant is defined as

$$\tilde{P}(s) := P(s) + \Delta P(s) \tag{15}$$

Let $\boldsymbol{P}_a$, the class of all allowable additive plant perturbations, be described by

$$\boldsymbol{P}_a = \Big\{\Delta P(s):\ \text{a) } \Delta P(s) \text{ is strictly proper rational} \tag{16a}$$

$$\text{b) } n_{\tilde{P}}^{+} = n_P^{+} \tag{16b}$$

$$\text{c) } \bar{\sigma}\big[\Delta P(j\omega)\big] < \zeta(\omega)\,\forall\,\omega \in \boldsymbol{R}_+\Big\} \tag{16c}$$

where $\zeta(\omega)$ is a given tolerance function, n_P^+ is the number of the unstable poles of $P(s)$, and $\boldsymbol{R}_+$ is the non-negative real line. Then we have the following lemma.

Lemma 2.1.[30,31] Consider the system $\mathcal{C}(P,K)$ shown in Eq. (13) and let $\boldsymbol{P}_a$ be described by Eqs. (15) and (16). If the nominal system $\mathcal{C}(P,K)$ is internally stable, then $\forall \Delta P \in \boldsymbol{P}_a$, $\mathcal{C}(\tilde{P},K)$ is internally stable if and only if

$$\bar{\sigma}\big[K(I-PK)^{-1}(j\omega)\big] < 1/\zeta(\omega) \quad \forall\ \omega \in \boldsymbol{R}_+ \tag{17}$$

Multiplicative Perturbations at the Output

The perturbed plant is defined as

$$\tilde{P}(s) := \big[I + M(s)\big]P(s) \tag{18}$$

Let $\boldsymbol{P}_{mo}$, the class of all allowable output multiplicative plant perturbations, be described by

$$\boldsymbol{P}_{mo} = \Big\{M(s):\ \text{a) } M(s) \text{ is proper rational} \tag{19a}$$

$$\text{b) } n_{\tilde{P}}^{+} = n_P^{+} \tag{19b}$$

$$\text{c) } \bar{\sigma}\big[M(j\omega)\big] < \zeta(\omega)\ \forall\,\omega \in \boldsymbol{R}_+\Big\} \tag{19c}$$

where $\zeta(\omega)$ is a given tolerance function, n_P^+ is the number of the unstable poles of $P(s)$, and $\boldsymbol{R}_+$ is the non-negative real line. Then we have the following lemma.

Lemma 2.2.[30,31] Consider the system $\mathcal{C}(P,K)$ shown in Eq. (13) and let $\boldsymbol{P}_{mo}$ be described by Eqs. (18) and (19). If the nominal system $\mathcal{C}(P,K)$ is internally stable, then $\forall M \in \boldsymbol{P}_{mo}$, $\mathcal{C}(\tilde{P},K)$ is internally stable if and only if

$$\bar{\sigma}\big[PK(I-PK)^{-1}(j\omega)\big] < 1/\zeta(\omega) \quad \forall\ \omega \in \boldsymbol{R}_+ \tag{20}$$

Multiplicative Perturbations at the Input

The perturbed plant is defined as

$$\tilde{P}(s) := P(s)\big[I + M(s)\big] \tag{21}$$

Let $\boldsymbol{P}_{mi}$, the class of all allowable input multiplicative plant perturbations, be described by

$$\boldsymbol{P}_{mi} = \Big\{M(s):\ \text{a) } M(s) \text{ is proper rational} \tag{22a}$$

$$\text{b) } n_{\tilde{P}}^{+} = n_P^{+} \tag{22b}$$

$$\text{c) } \bar{\sigma}\big[M(j\omega)\big] < \zeta(\omega)\quad \forall\ \omega \in \boldsymbol{R}_+\Big\} \tag{22c}$$

where $\zeta(\omega)$ is a given tolerance function, n_P^+ is the number of the unstable poles of $P(s)$, and $\boldsymbol{R}_+$ is the non-negative real line. Then we have the following lemma.

Lemma 2.3.[30,31] Consider the system $\mathcal{C}(P,K)$ shown in Eq. (13) and let $\boldsymbol{P}_{mi}$ be described by Eqs. (21) and (22). If the nominal system $\mathcal{C}(P,K)$ is internally stable, then $\forall M \in \boldsymbol{P}_{mi}$, $\mathcal{C}(\tilde{P},K)$ is internally stable if and only if

$$\bar{\sigma}\big[KP(I-KP)^{-1}(j\omega)\big] < 1/\zeta(\omega) \quad \forall\ \omega \in \boldsymbol{R}_+ \tag{23}$$

Note that Lemmas 2.1–2.3 are related to the issue of system robust stability. To maintain the internal stability under the prescribed perturbations, the complementary sensitivity functions $K(I-PK)^{-1}(s)$, $PK(I-PK)^{-1}(s)$, and $KP(I-KP)^{-1}(s)$ have to meet certain magnitude requirements [see Eqs. (17), (20), and (23)]. It is equivalent to express these requirements by using H^∞ norm:

$$\|W_2K(I-PK)^{-1}\|_\infty < 1 \tag{24a}$$

$$\|W_2PK(I-PK)^{-1}\|_\infty < 1 \tag{24b}$$

$$\|W_2KP(I-KP)^{-1}\|_\infty < 1 \tag{24c}$$

where $W_2(s)$ is a weighting matrix used to normalize specifications and can be expressed by

$$W_2(s) = w_2(s)I \tag{24d}$$

where I is an identity matrix, and $w_2(s)$ is a scale valued stable rational function with $|w_2(j\omega)| = \zeta(\omega)$ in Eqs. (16c), (19c), and (22c). These inequalities imply that the smaller the H^∞ norm of the complementary function, the better the robust stability of the system. Choosing the weighting matrix depends on the real-world environment. This will be discussed later.

Control-Input Constraint

To avoid the control input being to large, a control-input constraint is necessary. It is easy to see the $K(I-PK)^{-1}$ is the transfer function from v to u. Thus, the robust stability constraint (24a) also limits the maximal control-input energy. In other words, Eq. (24a) can be used to achieve the robust stability and the control-input energy constraint at the same time.

Standard H^∞ Optimization Problem

Most control problems can be formulated as the following standard H^∞ optimization problem. In the standard H^∞ optimization problem formulation, the system representation is rearranged as follows:

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{matrix} \\ p_1 \\ p_2 \end{matrix}\overset{\quad m_1 \qquad\quad m_2}{\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}} \begin{bmatrix} v(s) \\ u(s) \end{bmatrix} := G(s)\begin{bmatrix} v(s) \\ u(s) \end{bmatrix} \tag{25}$$

where $G_{11}(s) \in \boldsymbol{R}(s)^{p_1\times m_1}$, $G_{12}(s) \in \boldsymbol{R}(s)^{p_1\times m_2}$, $G_{21}(s) \in \boldsymbol{R}(s)^{p_2\times m_1}$, and $G_{22}(s) \in \boldsymbol{R}(s)^{p_2\times m_2}$. $\boldsymbol{R}(s)^{p\times q}$ is the set of $p\times q$ proper rational matrices with real coefficients. In Eq. (25), z, y, v, and u are the controlled ouput, the measured output, the exogenous input, and the control input, respectively. The controlled output vector z usually includes the error signal and a weighted control input. The exogenous input v contains the disturbances, noises, and commands. The measured output vector y consists of all of the signals that can be measured and available for feedback. Through the control input u the behavior of the system can be modified. The vector y will be used as the input to a controller $K(s)$, and the output of $K(s)$ wil be connected to the control input u; i.e.,

$$u(s) = K(s)y(s) \tag{26}$$

The standard H^∞ optimization problem consists of finding a proper controller $K(s)$ so that the closed-loop system is internally stable and $\|\mathfrak{F}_\ell(G,K)\|_\infty$ is minimized, where

$$\mathfrak{F}_\ell(G,K)(s) = G_{11}(s) + G_{12}(s)K(s)\big[I - G_{22}(s)K(s)\big]^{-1}G_{21}(s) \tag{27}$$

That is, $\mathfrak{F}_\ell(G,K)(s)$ is the transfer function of the closed-loop system from v to z.

Let

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{28}$$

be a state-space realization of the generalized plant $G(s)$ and $A \in R^{n\times n}$.

According to the dimensions of $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$, and $G_{22}(s)$, the standard H^∞ optimization problem has the following situations to consider: 1) $p_1 > m_2$, $p_2 < m_1$; 2) $p_1 \le m_2$, $p_2 < m_1$; 3) $p_1 > m_2$, $p_2 \ge m_1$; and 4) $p_1 \le m_2$, $p_2 \ge m_1$. Situation 1 is referred to as the four-block H^∞ optimization problem. Situations 2 and 3 are referred to as the two-block H^∞ optimization problem. Situation 4 is referred to as the one-block H^∞ optimization problem.

Glover-Doyle State-Space Formulas

In Ref. 18 Glover and Doyle assume that the realization of the generalized plant $G(s)$ is given by Eq. (28) with the following assumptions:

1) (A,B_2) is stabilizable, and (C_2,A) is detectable.
2) Rank $D_{12} = m_2$, and rank $D_{21} = p_2$.
3) $D_{12} = [0 \quad I]^T$, $D_{21} = [0 \quad I]$, and D_{11} is partitioned as

$$\begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix}$$

with $D_{1122} \in R^{p_2\times m_2}$.

4) $D_{22} = 0$ (this can be removed; for details refer to Ref. 18).
5)

$$\operatorname{rank}\begin{bmatrix} j\omega I - A & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2, \qquad \forall\, \omega \in R$$

6)

$$\operatorname{rank}\begin{bmatrix} j\omega I - A & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2, \qquad \forall\, \omega \in R$$

Note that assumptions 5 and 6 imply that $G_{12}(s)$ and $G_{21}(s)$ have no invariant zeros on the $j\omega$ axis. Define two Hamiltonian matrices as follows:

$$H_\infty(\gamma) := \begin{bmatrix} A & 0 \\ -C_1^TC_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^TD_{1\bullet} \end{bmatrix} R^{-1}\left[D_{1\bullet}^TC_1 \quad B^T\right] \tag{29a}$$

and

$$J_\infty(\gamma) := \begin{bmatrix} A^T & 0 \\ -B_1B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1D_{\bullet 1}^T \end{bmatrix} \tilde{R}^{-1}\left[D_{\bullet 1}B_1^T \quad C\right] \tag{29b}$$

where

$$D_{1\bullet} = \left[D_{11} \quad D_{12}\right], \qquad D_{\bullet 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \tag{29c}$$

and

$$R = D_{1\bullet}^TD_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{R} = D_{\bullet 1}D_{\bullet 1}^T - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix} \tag{29d}$$

Then the following theorem shows an easy way to construct a suboptimal stabilizing controller so that $\|\mathfrak{F}_\ell(G,K)\|_\infty < \gamma$, where $\mathfrak{F}_\ell(G,K)$ is the closed-loop transfer matrix from v to z.

Theorem 2.4.[18] There exists a stabilizing controller such that $\|\mathfrak{F}_\ell(G,K)\|_\infty < \gamma$ if and only if the following three conditions hold:

$$\text{(i)} \quad \gamma > \max\left(\bar{\sigma}\left[D_{1111} \quad D_{1112}\right],\ \bar{\sigma}\left[D_{1111}^T \quad D_{1121}^T\right]\right) \tag{30a}$$

$$\text{(ii)} \quad H_\infty(\gamma) \in \text{dom(Ric)} \quad \text{and} \quad X(\gamma) := \text{Ric}\left[H_\infty(\gamma)\right] \ge 0$$
$$J_\infty(\gamma) \in \text{dom(Ric)} \quad \text{and} \quad Y(\gamma) := \text{Ric}\left[J_\infty(\gamma)\right] \ge 0 \tag{30b}$$

$$\text{(iii)} \quad \rho\left[X(\gamma)Y(\gamma)\right] < \gamma^2 \tag{30c}$$

Moreover, when these conditions hold, one such controller is

$$K_{\text{sub}}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \tag{31a}$$

where

$$\hat{D} = -D_{1121}D_{1111}^T(\gamma^2 I - D_{1111}D_{1111}^T)^{-1}D_{1112} - D_{1122} \tag{31b}$$

$$\hat{C} = \{F_2 - \hat{D}(C_2 + F_{12})\}Z \tag{31c}$$

$$\hat{B} = -H_2 + (B_2 + H_{12})\hat{D} \tag{31d}$$

$$\hat{A} = A + HC + (B_2 + H_{12})\hat{C} \tag{31e}$$

$$Z := (I - \gamma^{-2}YX)^{-1} \tag{31f}$$

$$F^T = \left[F_{11}^T \quad F_{12}^T \quad F_2^T\right] = -(XB + C_1^TD_{1\bullet})R^{-1} \tag{31g}$$

$$H = \left[H_{11} \quad H_{12} \quad H_2\right] = -(YC^T + B_1D_{\bullet 1}^T)\tilde{R}^{-1} \tag{31h}$$

and $F_{11} \in R^{(m_1-p_2)\times n}$, $F_{12} \in R^{p_2\times n}$, $F_2 \in R^{m_2\times n}$, $H_{11} \in R^{n\times(p_1-m_2)}$, $H_{12} \in R^{n\times m_2}$, $H_2 \in R^{n\times p_2}$.

In Theorem 2.4, condition (ii) means that there exist positive-semidefinite stabilizing solutions X and Y to the algebraic Riccati equations corresponding to the Hamiltonians $H_\infty(\gamma)$ and $J_\infty(\gamma)$, respectively.[18] Condition (iii) means that the spectral radius of XY is less than γ^2.

Theorem 2.4 shows an easy state-space approach to construct a stabilizing suboptimal controller such that $\|\mathfrak{F}_\ell(G,K)\|_\infty < \gamma$. The order of the suboptimal controller can be the same as that of the plant $G(s)$. The major computation involved is the solution of two H^∞ Riccati equations that are easy to solve if solutions exist.

The GD approach is a breakthrough in the solution of the H^∞ optimization problem. Theorem 2.4 can be used to construct optimal or suboptimal controllers for one-, two-, and four-block plants. When an optimal controller is designed by the GD approach, however, one needs to compute the optimal H^∞ norm first. Algorithms for computing the optimal H^∞ norm and the construction of an optimal H^∞ controller will be discussed in Sec. IV. Section III explains how to formulate a robust control problem as a standard H^∞ optimization problem.

III. Formulation of H^∞ Optimization Problems

As mentioned in the previous section, many control problems can be formulated as the standard H^∞ optimization problem. For the purpose of demonstration, two examples are given in the following. The first is a mixed-sensitivity optimization problem to be formulated as a two-block H^∞ optimization problem: the second is a disturbance reduction problem with measurement noise that turns out to be a four-block problem.

Mixed-Sensitivity Optimization Problem

As mentioned earlier, for the feedback control system in Eq. (13) a smaller $\|(I-PK)^{-1}\|_\infty$ means a better disturbance attenuation, whereas a smaller $\|PK(I-PK)^{-1}\|_\infty$ implies a better robust stability. Unfortunately, the H^∞ norms of $(I-PK)^{-1}$ and $PK(I-PK)^{-1}$ may not be made small at the same time. If we make one of them smaller, the other will become larger. To have a tradeoff between these two quantities, Kwakernaak[12] formulated the mixed-sensitivity problem as the problem of finding a controller $K(s)$ that stabilizes the closed-loop system and minimizes $\|\Phi\|_\infty$, where Φ is given by

$$\Phi = \begin{bmatrix} W_1(I-PK)^{-1} \\ W_2PK(I-PK)^{-1} \end{bmatrix} \tag{32}$$

W_1 and W_2 are weighting matrices chosen by the designer according to the specific situation; i.e., they depend on the characters of the disturbances and system uncertainties. Usually, the disturbances occur most likely at a low frequency; therefore, $W_1(s)$ is chosen to be a low-pass filter to emphasize the error energy at low frequency. The plant uncertainty is also frequency dependent: The higher the frequency, the larger the uncertainties. Hence, $W_2(s)$ is usually chosen to be an improper transfer function [but $W_2P(s)$ has to be a proper transfer function], which is analytic in the closed right half plane. In the following we assume that $W_1(s)$ is strictly proper, $W_2(s)$ is a polynomial such that $W_2P(s)$ remains proper, and both of them are analytic in the closed right half plane.

The problem of finding a $K(s)$ that stabilizes the closed-loop system and minimizes $\|\Phi\|_\infty$ can be rearranged into the standard H^∞ optimization problem. Consider the following system:

$$\begin{bmatrix} z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} W_1 & W_1P \\ 0 & W_2P \\ \hline I & P \end{array}\right] \begin{bmatrix} v \\ u \end{bmatrix} \tag{33a}$$

$$u = Ky \tag{33b}$$

It is easy to show that the matrix Φ defined by Eq. (32) is just the transfer function from v to $[z_1^T \quad z_2^T]^T$ of the closed-loop system (33). Comparing Eq. (33a) with Eq. (25), we can see that

$$G_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}, \qquad G_{12} = \begin{bmatrix} W_1P \\ W_2P \end{bmatrix}$$

$$G_{21} = I, \qquad G_{22} = P \tag{34}$$

If P, W_2P, and W_1 have the following state-space realizations:

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \quad W_2P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_{w_2} & D_{w_2} \end{array}\right]$$

$$W_1 = \left[\begin{array}{c|c} A_{w_1} & B_{w_1} \\ \hline C_{w_1} & D_{w_1} \end{array}\right] \tag{35}$$

Then the generalized plant $G(s)$ has a state-space realization as shown in Eq. (30), with

$$A = \begin{bmatrix} A_p & 0 \\ B_{w_1}C_p & A_{w_1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ B_{w_1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ B_{w_1}D_p \end{bmatrix}$$

$$C_1 = \begin{bmatrix} D_{w_1}C_p & C_{w_1} \\ C_{w_2} & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} D_{w_1} \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} D_{w_1}D_p \\ D_{w_2} \end{bmatrix}$$

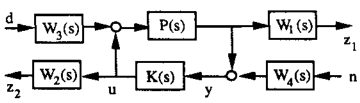

Fig. 1 Disturbance attenuation problem.

$$C_2 = [C_p \quad 0], \qquad D_{21} = I, \qquad D_{22} = D_p \tag{36}$$

Note that, because W_2 is a polynomial, the A matrix for W_2P is the same as that for P.

Disturbance Reduction Problem

Consider the feedback system shown in Fig. 1. $P(s)$ is a given plant, $W_i(s)$ $(i=1,2,3,4)$ are weighting matrices, and $K(s)$ is the controller to be designed. The disturbance and noise are the outputs of W_3 and W_4 driven by d and n, respectively. The symbol z_1 is the weighted error response, and z_2 is the weighted control input. Let $z^T=[z_1^T \; z_2^T]^T$, $v^T=[d^T \; n^T]^T$, and assume that v is unknown but with its energy bounded by unity. The objective is to find a controller $K(s)$ that stabilizes the closed-loop system and minimizes the worst $\|z\|_2$, i.e., minimizes the H^∞ norm of T_{zv}, the closed-loop transfer function from v to z. T_{zv} is given by

$$T_{zv} = \begin{bmatrix} W_1P(I-KP)^{-1}W_3 & W_1PK(I-PK)^{-1}W_4 \\ W_2KP(I-KP)^{-1}W_3 & W_2K(I-PK)^{-1}W_4 \end{bmatrix} \tag{37}$$

Note that $W_1PK(I-PK)^{-1}W_4$ and $W_2KP(I-KP)^{-1}W_3$ are the output and input complementary sensitivity functions, respectively. Their H^∞ norms indicate the stability robustness of the closed-loop system for the multiplicative plant uncertainty introduced at the output and input, respectively. $W_2K(I-PK)^{-1}W_4$ is the control complementary sensitivity function whose H^∞ norm indicates the stability robustness of the closed-loop system for additive plant uncertainty. Hence, reducing $\|T_{zv}\|_\infty$ will also improve the robust stability of the closed-loop system.

It is easy to verify that the generalized plant of the system can be expressed as

$$\begin{bmatrix} z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cc|c} W_1PW_3 & 0 & W_1P \\ 0 & 0 & W_2 \\ \hline PW_3 & W_4 & P \end{array}\right] \begin{bmatrix} d \\ n \\ \hline u \end{bmatrix} \tag{38}$$

That is,

$$G_{11} = \begin{bmatrix} W_1PW_3 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G_{12} = \begin{bmatrix} W_1P \\ W_2 \end{bmatrix}$$

$$G_{21} = [PW_3 \quad W_4], \qquad G_{22} = P \tag{39}$$

If P, W_i $(i=1,2,3,4)$ have state-space realizations as follows:

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right], \qquad W_i = \left[\begin{array}{c|c} A_{wi} & B_{wi} \\ \hline C_{wi} & D_{wi} \end{array}\right] \tag{40}$$

where $i=1,2,3,4$, then the generalized plant $G(s)$ has a state-space realization as shown in Eq. (28), with

$$A = \begin{bmatrix} A_p & 0 & 0 & B_pC_{w_3} & 0 \\ B_{w_1}C_p & A_{w_1} & 0 & B_{w_1}D_pC_{w_3} & 0 \\ 0 & 0 & A_{w_2} & 0 & 0 \\ 0 & 0 & 0 & A_{w_3} & 0 \\ 0 & 0 & 0 & 0 & A_{w_4} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} B_pD_{w_3} & 0 \\ B_{w_1}D_pD_{w_3} & 0 \\ 0 & 0 \\ B_{w_3} & 0 \\ 0 & B_{w_4} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} B_p \\ B_{w_1}D_p \\ B_{w_2} \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} D_{w_1}C_p & C_{w_1} & 0 & D_{w_1}D_pC_{w_3} & 0 \\ 0 & 0 & C_{w_2} & 0 & 0 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} D_{w_1}D_pD_{w_3} & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} D_{w_1}D_p \\ D_{w_2} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} C_p & 0 & 0 & D_pC_{w_3} & C_{w_4} \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} D_pD_{w_3} & D_{w_4} \end{bmatrix}, \qquad D_{22} = D_p \tag{41}$$

In Eq. (41), $\{A, B, C, D\}$ is the state-space representation for the generalized plant $G(s)$. However, it is possible to reduce the order of the system before going to the next step.

IV. Solution to H^∞ Optimization Problems

Once the H^∞ optimization problem is formulated, the next step is to compute the optimal H^∞ norm and then to construct an optimal or suboptimal H^∞ controller. It is reasonable to assume that D_{12} and D_{21} are of full column rank and of full row rank, respectively. Before we use the GD formulas, the inputs and outputs need to be scaled so that the assumptions $D_{21} = [0 \;\; I]$ and $D_{12} = [0 \;\; I]^T$ are satisfied.

Scaling of D_{12} and D_{21}

Step 1. Find a singular value decomposition for D_{12}:

$$D_{12} = [U_{121} \quad U_{122}] \begin{bmatrix} \Sigma_{12} \\ 0 \end{bmatrix} V_{12}^T \tag{42}$$

Let

$$L_{12} = \begin{bmatrix} U_{122}^T \\ U_{121}^T \end{bmatrix}$$

and

$$R_{12} = V_{12}\Sigma_{12}^{-1}$$

Then we have

$$L_{12}D_{12}R_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Step 2. Find the singular value decomposition for D_{21}:

$$D_{21} = U_{21}[\Sigma_{21} \quad 0] \begin{bmatrix} V_{211}^T \\ V_{212}^T \end{bmatrix} \tag{43}$$

Let $L_{21} = \Sigma_{21}^{-1}U_{21}^T$ and $R_{21} = [V_{212} \quad V_{211}]$. Then we have $L_{21}D_{21}R_{21} = [0 \;\; I]$.

Step 3. Scale the inputs and outputs. Define new inputs $\tilde{v}$, $\tilde{u}$ and new outputs $\tilde{z}$, $\tilde{y}$ as follows:

$$v = R_{21}\tilde{v}, \qquad u = R_{12}\tilde{u}$$

$$\tilde{z} = L_{12}z, \qquad \tilde{y} = L_{21}y \tag{44}$$

Then the new generalized plant $\tilde{G}(s)$, i.e., the transfer function from $[\tilde{v}^T \quad \tilde{u}^T]^T$ to $[\tilde{z}^T \quad \tilde{y}^T]^T$, has the following state-space realization:

$$\tilde{G}(s) = \left[\begin{array}{c|cc} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right] \tag{45}$$

where

$$\tilde{A} = A, \qquad \tilde{B}_1 = B_1R_{21}, \qquad \tilde{B}_2 = B_2R_{12}$$

$$\tilde{C}_1 = L_{12}C_1, \qquad \tilde{D}_{11} = L_{12}D_{11}R_{21}, \qquad \tilde{D}_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\tilde{C}_2 = L_{21}C_2, \qquad \tilde{D}_{21} = [0 \quad I], \qquad \tilde{D}_{22} = L_{21}D_{22}R_{12}$$

If an optimal or suboptimal controller for the scaled generalized plant $\tilde{G}(s)$ is

$$\tilde{K}(s) = \left[\begin{array}{c|c} \tilde{A}_k & \tilde{B}_k \\ \hline \tilde{C}_k & \tilde{D}_k \end{array}\right] \tag{46}$$

then the corresponding controller for its original generalized plant will be

$$K(s) = \left[\begin{array}{c|c} \tilde{A}_k & \tilde{B}_kL_{21} \\ \hline R_{12}\tilde{C}_k & R_{12}\tilde{D}_kL_{21} \end{array}\right] \tag{47}$$

Note that $\|T_{\tilde{z}\tilde{v}}\|_\infty = \|T_{zv}\|_\infty$.

Computation of the Optimal H^∞ Norm

In utilizing these GD formulas to design an optimal H^∞ controller, the most computationally demanding work is the computation of the optimal H^∞ norm that requires iteration.

According to the GD theorem, we can see in Eqs. (29a) and (29b) that both X and Y are functions of γ. Moreover, the problem of finding the optimal H^∞ norm, denoted γ_o, is equivalent to finding the infimum γ such that all three conditions in Eq. (30) hold. To make it more precise, we give the following definitions.

Definition 4.1.

a) $\alpha_x := \inf\{\gamma : \gamma \in \boldsymbol{R}_+ \text{ and } X(\gamma) \text{ exists}\}$

$\alpha_y := \inf\{\gamma : \gamma \in \boldsymbol{R}_+ \text{ and } Y(\gamma) \text{ exists}\}$

b) $\beta_x := \inf\{\gamma : \gamma \in \boldsymbol{R}_+$ and $X(\gamma)$ is positive semidefinite$\}$

$\beta_y := \inf\{\gamma : \gamma \in \boldsymbol{R}_+$ and $Y(\gamma)$ is positive semidefinite$\}$

c) $\alpha := \max\{\alpha_x, \alpha_y\}$

$\beta := \max\{\beta_x, \beta_y\}$

where $X(\gamma)$ and $Y(\gamma)$ are the stabilizing solutions to Riccati equations in Eq. (30b).

Figure 2 shows the domain of the Riccati solutions. It is obvious that $\gamma_o \in [\beta, +\infty)$. It is possible for β to be γ_o, especially when β and α are identical; however, with very few exceptions, $\gamma_o \in (\beta, +\infty)$, which implies that γ_o is the solution to $\rho[X(\gamma)Y(\gamma)] = \gamma^2$.

Figure 2 implies that the problem of finding γ_o is actually that of either searching for the intersection point of $\rho[X(\gamma)Y(\gamma)]$ with γ^2 inside $(\beta, +\infty)$ or computing the boundary point β. An efficient algorithm to compute γ_o can be found in Ref. 25, where the convex properties of Riccati solutions were employed.

Construction of an Optimal H^∞ Controller

From the GD formulas in Theorem 2.4, a suboptimal H^∞ controller can easily be constructed. However, as γ approaches to the optimum, the formulas in Eqs. (31) may not remain well defined when $\gamma_o \in (\beta, +\infty)$ (see Fig. 2 for reference). To eliminate this numerical difficulty, Safonov et al.[20] rederived the optimal controller formulas in a descriptor form (or generalized state-space representation).

The GD formulas in Eqs. (31a-31h) can also be written in a descriptor form after slight rearrangement. When γ reaches the optimum, $\gamma_o \in (\beta, +\infty)$, which satisfies $\gamma_o^2 = \rho[X(\gamma_o)Y(\gamma_o)]$, the matrix Z in Eq. (31f) will become infin-

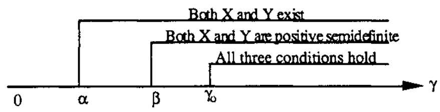

Fig. 2 Domain of Riccati solutions.

ity since the matrix $I - \gamma_o^{-2}Y(\gamma_o)X(\gamma_o)$ is singular. If we try to apply formulas (31a–31h) directly to construct an optimal H^∞ controller, a numerical difficulty will arise in the implementation of the $\hat{A}$ and $\hat{C}$ matrices. We will rearrange these formulas so that an optimal H^∞ controller can be constructed without any numerical difficulty.

The dual system of the realization in Eq. (31a) can be easily rewritten in a descriptor form. The state equation (generalized state equation) of the descriptor representation can be split into two sets of equations: one involves the first derivatives of some state variables, and the other consists of algebraic equations. The state variables that have no derivatives in the equations can be eliminated; then we have a lower-order state-space representation for the dual system. The dual of the dual system is identical to the original system; therefore, we have an optimal H^∞ controller as follows:

$$K_{\text{opt}}(s) = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] \tag{48}$$

where

$$A_c = \left[V_1^T A_D U_1 - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1\right]\Sigma_1^{-1} \tag{49a}$$

$$B_c = V_1^T B_D - V_1^T A_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{49b}$$

$$C_c = \left[C_D U_1 - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T A_D U_1\right]\Sigma_1^{-1} \tag{49c}$$

$$D_c = \hat{D} - C_D U_2 (V_2^T A_D U_2)^\dagger V_2^T B_D \tag{49d}$$

In Eq. (49a–d), the superscript † means pseudo-inverse and A_D, B_D, C_D are given as follows:

$$B_D = -H_2 + (B_2 + H_{12})\hat{D} \tag{50a}$$

$$C_D = F_2 - \hat{D}(C_2 + F_{12}) \tag{50b}$$

$$A_D = (B_2 + H_{12})C_D + (A + HC)E_D \tag{50c}$$

where

$$E_D = I - \gamma_o^{-2}X(\gamma_o)Y(\gamma_o) \tag{50d}$$

and Σ_1, U_1, U_2, V_1, and V_2 are obtained from the singular value decomposition of E_D, i.e.,

$$E_D = [U_1 \quad U_2]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \tag{51}$$

V. Illustrative Examples

Two illustrative examples are included in this section. Example 1 is a mixed-sensitivity problem formulated as a two-block H^∞ optimization problem. Example 2 is a simple disturbance reduction problem with measurement noise. It is formulated as a four-block H^∞ optimization problem.

Example 1

This example is from Yeh et al.[32] and Dickman,[33] where the real parameter uncertainty problem was discussed. We use this example to illustrate the mixed-sensitivity problem described in Sec. III. The plant is given as

$$P(s) = \begin{bmatrix} \dfrac{1}{s-1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+2} \end{bmatrix}$$

The filters are chosen as

$$W_1(s) = \left\{(s+100)/\left[100(s+1)\right]\right\} I_2$$

which is a low-pass filter, and $W_2(s) = 0.1s + 1$, which is used to emphasize the high-frequency uncertainties. Then we have state-space realizations for P, W_2P, and W_1 as follows [refer to Eq. (35)]:

$$A_p = \text{diag}(1, -1, -2, -1), \qquad B_p = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \qquad D_p = 0$$

$$A_{w_1} = -0.1I_2, \quad B_{w_1} = 0.099I_2, \quad C_{w_1} = I_2, \quad D_{w_1} = 0.01I_2$$

$$C_{w_2} = \begin{bmatrix} 1.1 & 0 & 0 & 0.9 \\ 0 & 0.9 & 0.8 & 0 \end{bmatrix}, \qquad D_{w_1} = 0.1I_2$$

From Eq. (36), a realization of the generalized plant can be constructed easily. Then by use of the algorithm described by Li and Chang,[25] the optimal H^∞ norm of the system can be obtained ($\gamma_o = 1.16824$). Furthermore, Eqs. (48-51) could be used to construct the optimal controller $K(s)$. The $\bar{\sigma}$ plot of complementary sensitivity function $PK(I - PK)^{-1}$ is shown in Fig. 3. As shown in Eqs. (18–20), we can see that, if we have a multiplicative perturbation introduced at the output and if the norm of perturbation is below the plot in Fig. 4, then the system will remain internally stable.

Figure 5 is related to the disturbance attenuation problem. We can see that the maximal value is ~2.1, which indicates the

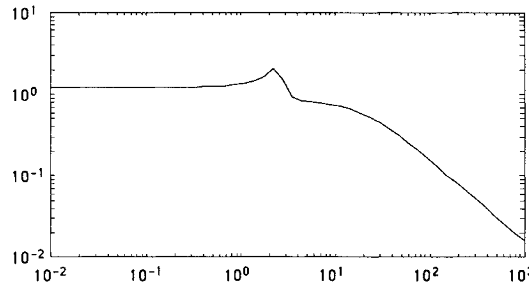

Fig. 3 $\bar{\sigma}$ **plot of complementary sensitivity function** $PK(I - PK)^{-1}$.

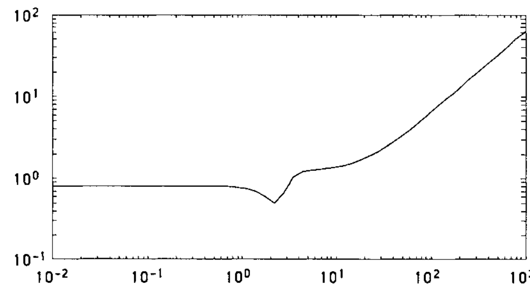

Fig. 4 Perturbation toleration.

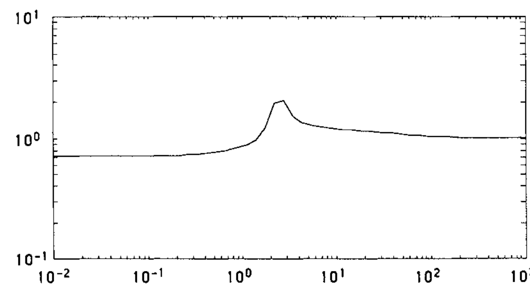

Fig. 5 $\bar{\sigma}$ **plot of sensitivity function** $(I - PK)^{-1}$.

Table 1 Convergence of the algorithm

Iteration	γ_n	Accuracy
0	100	$9.535e+01$
1	$4.648e+00$	$8.403e-01$
2	$4.734e+00$	$9.440e-04$
3	$4.734e+00$	$1.115e-09$
4	$4.734e+00$	$7.105e-15$

output energy for the "worst disturbance" with unit energy. However, if the disturbance happens in low frequency, then the worst output energy is just about 0.75.

Example 2

The following is a simple four-block H^∞ optimization problem. Consider the system in Fig. 1 with

$$P(s) = \frac{1}{s-2}, \quad W_3(s) = \frac{1}{s+1}, \quad W_1(s) = W_2(s) = W_4(s) = 1$$

State-space realizations for P and W_3 are

$$P(s) = \left[\begin{array}{c|c} 2 & 1 \\ \hline 1 & 0 \end{array}\right], \qquad W_3(s) = \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right]$$

From Eq. (37) a realization of the generalized plant can be constructed as

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|cc:c} -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 1 & 0 & 1 & 0 \end{array}\right]$$

The data in Table 1 show that four iterations are needed to reach the optimum, $\gamma_o = 4.734160476390407$, with accuracy better than 10^{-14}.

By use of formulas (48-51), an optimal H^∞ controller is constructed as follows:

$$K_{\text{opt}}(s) = \left[\begin{array}{c|c} -0.87542 & -0.13925 \\ \hline 4.42042 & -4.73416 \end{array}\right]$$

where the H^∞ norm of the closed-loop system equals γ_o. Note that the optimal controller has a direct feedthrough term and thus has an infinite bandwidth. If we choose $\gamma = 4.8$, which is about 1.4% higher than γ_o, we have a suboptimal controller,

$$K_{\text{opt}}(s) = \left[\begin{array}{cc|c} -8.67072e-01 & 1.32928e-01 & -1.38959e-01 \\ -1.38320e+01 & -1.52323e+02 & 4.73733e+00 \\ \hline -9.30025e+00 & -1.49792e+02 & 0 \end{array}\right]$$

which has a reasonable bandwidth and the closed-loop H^∞ norm, $\|T_{zv}\|_\infty < 4.8$, which is only 1.4% away from the optimal H^∞ norm.

VI. Conclusion

The formulation of the standard H^∞ optimization problem, an easy way of constructing a state-space realization of the generalized plant, an efficient algorithm for computing the optimal H^∞ norm, and a modified version of GD formulas for constructing an optimal controller were addressed in this paper. No numerical difficulty will arise in constructing an H^∞ optimal controller if we are allowed a proper controller. In most applications we may like to have a strictly proper controller with a limited bandwidth. In this case a tradeoff between the H^∞ performance and the bandwidth should be made by degrading the H^∞ norm from its optimum in order to reduce the controller bandwidth.

Acknowledgment

This research was funded in part by AFOSR Contract F33615-90-C-3600 and in part by NASA Langley Research Center Grant NAG-1-1102.

References

[1]Zames, G., "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses," *IEEE Transactions on Automatic Control,* Vol. AC-26, No. 2, 1981, pp. 301-320.

[2]Francis, B. A., and Zames, G., "On H^∞-Optimal Sensitivity Theory for SISO Feedback Systems," *IEEE Transactions on Automatic Control,* Vol. AC-29, No. 1, Jan. 1984, pp. 9-16.

[3]Francis, B. A., Helton, J. W., and Zames, G., "H^∞-Optimal Feedback Controllers for Linear Multivariable Systems," *IEEE Transactions on Automatic Control,* Vol. AC-29, No. 10, Oct. 1984, pp. 888-900.

[4]Chang, B., C., and Pearson, J. B., "Optimal Disturbance Reduction in Linear Multivariable Systems," *IEEE Transactions on Automatic Control,* Vol. AC-29, Oct. 1984, pp. 880-887.

[5]Safonov, M. G., and Verma, M. S., "L^∞-Sensitivity Optimization via Hankel-Norm Approximation," *Proceedings of the American Control Conference,* IEEE, San Francisco, CA, June 1983, pp. 1280-1283.

[6]Glover, K., "All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and Their L^∞-Error Bounds," *International Journal of Control,* Vol. 39, No. 6, 1984, pp. 1115-1193.

[7]Allison, A. C., and Young, N. J., "Numerical Algorithms for the Nevanlinna-Pick Problem," *Numerische Mathematik,* Vol. 42, 1983, pp. 125-145.

[8]Doyle, J., "Synthesis of Robust Controllers and Filters," *Proceedings of the 22nd Conference on Decision and Control,* IEEE, San Antonio, TX, Dec. 1983.

[9]Doyle, J., lectures notes, ONR/Honeywell Workshop on Advances in Multivariable Control, Minneapolis, MN, Oct. 1984.

[10]Chu, C.-C., "H^∞ Optimization and Robust Multivariable Control," Ph.D. Dissertation, Univ. of Minnesota, Minneapolis, MN, Sept. 1985.

[11]Kimura, H., "Robust Stabilization for a Class of Transfer Functions," *IEEE Transactions on Automatic Control,* Vol. AC-29, No. 9, Sept. 1984, pp. 788-793.

[12]Kwakernaak, H., "Minimax Frequency Domain Performance and Robustness Optimzation of Linear Feedback Systems," *IEEE Transactions on Automatic Control,* Vol. AC-30, No. 10, Oct. 1985, pp. 994-1004.

[13]Francis, B. A., *A Course in H^∞ Theory,* Springer-Verlag, Berlin, Heidelberg, 1987.

[14]Chang, B. C., and Pearson, J. B., "Iterative Computation of Minimal H^∞ Norm," *Proceedings of the 24th Conference on Decision and Control,* IEEE, Fort Lauderdale, FL, Dec. 1985, pp. 1307-1310.

[15]Chu, C.-C., Doyle, J., and Lee, E. B., "The General Distance Problem in H^∞ Optimal Control Theory," *International Journal of Control,* Vol. 44, No. 2, 1986, pp. 565-596.

[16]Jonckheere, E. A., and Juang, J. C., "Fast Computation on Achievable Feedback Performance in Mixed Sensitivity H_∞ Design," *IEEE Transactions on Automatic Control,* Vol. AC-32, No. 10, Oct. 1987,pp. 896-906.

[17]Chang, B. C., Banda, S. S., and McQuade, T. E., "Fast Iterative Computation of Optimal Two-Block H^∞ Norm," *IEEE Transactions on Automatic Control,* Vol. AC-34, No. 7, July 1989, pp. 738-743.

[18]Glover, K., and Doyle, J. C., "State Space Formulae for All Stabilizing Controllers that Satisfy an H^∞-Norm Bound and Relations to Risk Sensitivity," *Systems and Control Letters,* Vol. 11, 1988, pp. 167-172.

[19]Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Transactions on Automatic Control,* Vol. AC-34, No. 8, Aug. 1989, pp. 831–847.

[20]Safonov, M. G., Limebeer, D. J. N., and Chiang, R. Y., "Simplifying the H^∞ Theory via Loop-Shifting, Matrix-Pencil and Descriptor Concepts," *International Journal of Control,* Vol. 50, No. 6, 1989, pp. 2467–2488.

[21]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems,* Wiley, New York, 1972.

[22]Youla, D. C., Bongiorno, J. J., and Jabr, H. A., "Modern Wiener-Hopf Design of Optimal Controllers, Part II: The Multivariable Case," *IEEE Transactions on Automatic Control,* Vol. AC-21, 1976, pp. 319–338.

[23]Doyle, J., "Analysis of Feedback Systems with Structured Uncertainties," *IEEE Proceedings,* Vol. 129, Pt. D, No. 6, 1982, pp. 242–250.

[24]Doyle, J., "Structured Uncertainty in Control System Design," *Proceedings of the 24th Conference on Decision and Control,* Dec. 1985.

[25]Li, X. P., and Chang, B. C., "On Convexity of Riccati Solutions and Its Applications," *IEEE Transactions on Automatic Control* (to appear in March 1993).

[26]Misra, P., and Patel, R. V., "Minimal Order Generalized State Space Representation of Singular Systems," *Proceedings of the 1989 American Control Conference,* IEEE, Pittsburgh, PA, June 1989, pp. 2140–2145.

[27]Naylor, A. W., and Sell, G. R., *Linear Operator Theory in Engineering and Science,* Springer-Verlag, New York, 1982.

[28]Desoer, C. A., and Vidyasagar, M., *Feedback Systems: Input-Output Properties,* Academic, New York, 1975.

[29]Carlson, A. B., *Communication Systems: An Introduction to Signals and Noise in Electrical Communcation,* McGraw-Hill, New York, 1975.

[30]Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," *IEEE Transactions on Automatic Control,* Vol. AC-26, No. 2, Feb. 1981, pp. 4–16.

[31]Lehtomaki, N. A., Sandell, N. R., and Athans, M., "Robustness Results in Linear-Quadratic-Gaussian Based Multivariable Control Design," *IEEE Transactions on Automatic Control,* Vol. AC-26, No. 2, Feb. 1981, pp. 75–92.

[32]Yeh, H.-H., Banda, S. S., Heise, S. A., and Bartlett, A. C., "Robust Control Design with Real-Parameter Uncertainty and Unmodeled Dynamics," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 6, 1990, pp. 1117–1125.

[33]Dickman, A., "Method for Robust Design of Multivariable Feedback Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 5, 1987, pp. 500–503.